MATH 1104 LINEAR ALGEBRA

LECTURE NOTES

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(These Lecture Notes replace neither the Text Book nor the Lectures)

Part 5

- Complex Eigenvalues.

- Complex Eigenvectors.

COMPLEX EIGENVALUES and EIGENVECTORS

For a complex scalar λ,

$$\det(A - \lambda I) = 0 \Longleftrightarrow Ax = \lambda x$$

for some non-zero vector x in C^n.

λ is called a (complex) eigenvalue and x is called a (complex) eigenvector.

If x is a complex vector in C^n, then the vector $\overline{x}$, whose entries are the complex conjugates of the entries in x, is called the complex conjugate of x. Thus, if $x =$Re $x + i$ Im x, then $\overline{x} =$Re $x - i$ Im x.

Real matrices with complex eigenvalues:

Let A be a real $n \times n$ matrix and λ be an eigenvalue of A with a corresponding eigenvector x in C^n. Then $\overline{\lambda}$ is also an eigenvalue of A with the corresponding eigenvector $\overline{x}$:

$$\begin{aligned} Ax &= \lambda x \\ \overline{Ax} &= \overline{\lambda x} \\ \overline{A}\overline{x} &= \overline{\lambda}\overline{x} \\ A\overline{x} &= \overline{\lambda}\overline{x} \end{aligned}$$

If A is a real matrix, its complex eigenvalues occur in conjugate pairs.

Example: Let $A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$.
a) Find the eigenvalues of A, and a basis for each eigenspace in C^2.
b) Diagonalize A.
Solution:

$$A - \lambda I = \begin{bmatrix} 1-\lambda & -2 \\ 1 & 3-\lambda \end{bmatrix} \Longrightarrow \det(A - \lambda I) = \lambda^2 - 4\lambda + 5 = 0,$$

$$\lambda_{1,2} = \frac{4 \pm \sqrt{(-4)^2 - 4 \cdot 1 \cdot 5}}{2} = 2 \pm i.$$

$\underline{\lambda_1 = 2+i:}$

$$A-(2+i)I = \begin{bmatrix} -1-i & -2 \\ 1 & 1-i \end{bmatrix} \sim \begin{bmatrix} 1 & 1-i \\ 0 & 0 \end{bmatrix}.$$

$$\begin{cases} (A-(2+i)I)x = 0 \\ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_2 \begin{bmatrix} -1+i \\ 1 \end{bmatrix}. \end{cases}$$

$$v_1 = \begin{bmatrix} -1+i \\ 1 \end{bmatrix}, \quad E_{2+i} = \text{Span}\left\{ \begin{bmatrix} -1+i \\ 1 \end{bmatrix} \right\}.$$

$\underline{\lambda_2 = 2-i:}$

Since $\lambda_2 = \overline{\lambda_1}$, an eigenvector corresponding to λ_2 is

$$v_2 = \overline{v_1} = \begin{bmatrix} -1-i \\ 1 \end{bmatrix} \text{ and } E_{2-i} = \text{Span}\left\{ \begin{bmatrix} -1-i \\ 1 \end{bmatrix} \right\}.$$

b) $A = PDP^{-1}$, where

$$P = \begin{bmatrix} -1+i & -1-i \\ 1 & 1 \end{bmatrix}, D = \begin{bmatrix} 2+i & 0 \\ 0 & 2-i \end{bmatrix}.$$

Theorem: Let A be a real 2×2 matrix with a complex eigenvalue $\lambda = a - bi$ $(b \neq 0)$ and associated eigenvector v in C^2. Then,

$$A = PCP^{-1}, \text{ where } P = [\text{Re}v \quad \text{Im}v] \quad \text{and} \quad C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

Remark: Let $\lambda = a \pm ib$. Then,

$$r = |\lambda| = \sqrt{a^2 + b^2}, \cos\theta = \frac{a}{r}, \sin\theta = \frac{b}{r}.$$

θ is the angle between the positive x-axis and the ray from $(0,0)$ through (a,b).

$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = r \begin{bmatrix} a/r & -b/r \\ b/r & a/r \end{bmatrix} = \underbrace{\begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}}_{\text{scaling}} \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\text{rotation}}.$$

Example: The matrix $A = \begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$ is a real 2×2 matrix with a complex eigenvalue $\lambda = 2 - i$, $((a,b) = (2,1))$ and associated eigenvector

$$v = \begin{bmatrix} -1 - i \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}_{\text{Rev}} + i \underbrace{\begin{bmatrix} -1 \\ 0 \end{bmatrix}}_{\text{Imv}}.$$

By the above Theorem,

$$A = PCP^{-1} \text{ where } P = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}.$$

Remark: With the complex eigenvalue $\overline{\lambda} = 2 + i = 2 - (-1)i$,

$((a,b) = (2,-1))$ and associated eigenvector

$\overline{v} = \begin{bmatrix} -1 + i \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ in C^2:

$$A = PCP^{-1} \text{ where } P = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } C = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}.$$

Example: Let $A = \begin{bmatrix} 0 & 1 \\ -8 & 4 \end{bmatrix}$. Find an invertible matrix P and a matrix C of the form $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ such that $A = PCP^{-1}$.

Solution: $\det(A - \lambda I) = \lambda^2 - 4\lambda + 8 = 0 \Longleftrightarrow \lambda = 2 \pm 2i$.

$$\underline{\lambda_1 = 2 - 2i} \Longrightarrow (a, b) = (2, 2) \text{ and } C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 2 & 2 \end{bmatrix}.$$

$$A - (2 - 2i)I = \begin{bmatrix} -2 + 2i & 1 \\ -8 & 2 + 2i \end{bmatrix}.$$

$$(A - (2 - 2i)I)x = 0 \Longleftrightarrow x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{x_2}{4} \begin{bmatrix} 1 + i \\ 4 \end{bmatrix}.$$

$$v_1 = \begin{bmatrix} 1 + i \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} + i \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Longrightarrow P = \begin{bmatrix} \operatorname{Re} v_1 & \operatorname{Im} v_1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 4 & 0 \end{bmatrix}.$$

Remark 1: If you start with $\lambda_2 = 2 + 2i$, then

$$\lambda_2 = 2 + 2i = 2 - (-2)i \Longrightarrow (a, b) = (2, -2) \Longrightarrow C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ -2 & 2 \end{bmatrix}.$$

$$\lambda_2 = \overline{\lambda_1} \Longrightarrow v_2 = \overline{v_1} = \begin{bmatrix} 1 - i \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} + i \begin{bmatrix} -1 \\ 0 \end{bmatrix}.$$

Thus

$$P = \begin{bmatrix} \operatorname{Re} v_2 & \operatorname{Im} v_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 4 & 0 \end{bmatrix}.$$

For the matrices P and C, $A = PCP^{-1}$.

Example: Let $A = \begin{bmatrix} 0 & 3+4i \\ 3-4i & 0 \end{bmatrix}$.
Find the eigenvalues and corresponding eigenvectors of A.
Solution:

$$\det(A-\lambda I) = \begin{vmatrix} -\lambda & 3+4i \\ 3-4i & -\lambda \end{vmatrix} = \lambda^2 - 25 = 0 \Longrightarrow \lambda_1 = 5, \quad \lambda_2 = -5.$$

$\underline{\lambda_1 = 5:}$

$$A - 5I = \begin{bmatrix} -5 & 3+4i \\ 3-4i & -5 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{-3-4i}{5} \\ 0 & 0 \end{bmatrix}.$$

$$\begin{cases} (A-5I)x = 0 \\ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \dfrac{x_2}{5} \begin{bmatrix} 3+4i \\ 5 \end{bmatrix}. \\ v_1 = \begin{bmatrix} 3+4i \\ 5 \end{bmatrix} \\ E_5 = \operatorname{Span}\left\{ \begin{bmatrix} 3+4i \\ 5 \end{bmatrix} \right\}. \end{cases}$$

$\underline{\lambda_2 = -5:}$

$$A - (-5)I = \begin{bmatrix} 5 & 3+4i \\ 3-4i & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{3+4i}{5} \\ 0 & 0 \end{bmatrix}.$$

$$\begin{cases} (A+5I)x = 0 \\ x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \dfrac{x_2}{5} \begin{bmatrix} -(3+4i) \\ 5 \end{bmatrix}. \\ v_2 = \begin{bmatrix} -3-4i \\ 5 \end{bmatrix} \\ E_2 = \operatorname{Span}\left\{ \begin{bmatrix} -3-4i \\ 5 \end{bmatrix} \right\}. \end{cases}$$

Note that the matrix A has :

- complex entries,
- real eigenvalues,
- complex eigenvectors v_1 and v_2 such that $v_2 \neq \overline{v_1}$.

$$A = PDP^{-1} \quad \text{where} \quad P = \begin{bmatrix} 3+4i & -3-4i \\ 5 & 5 \end{bmatrix} \text{ and } D = \begin{bmatrix} 5 & 0 \\ 0 & -5 \end{bmatrix}.$$

Example: Let $A = \begin{bmatrix} i & 0 & 1 \\ 0 & i & 0 \\ 0 & 0 & i \end{bmatrix}$. If possible, diagonalize A.

Solution: $\lambda = i$ is the only eigenvalue of A.

$$A - iI = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$(A - iI)x = 0 \Longleftrightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

$$E_i = \text{Span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Since A is a 3×3 matrix and has only two linearly independent eigenvectors, A is not diagonalizable.

Example: Let $A = \begin{bmatrix} 2 & 0 & -4 \\ 0 & 1 & 0 \\ 2 & 0 & -2 \end{bmatrix}$. If possible, diagonalize A.

Solution:

$$\det(A-\lambda I) = \begin{vmatrix} 2-\lambda & 0 & -4 \\ 0 & 1-\lambda & 0 \\ 2 & 0 & -2-\lambda \end{vmatrix} = (1-\lambda)(\lambda^2+4) = 0 \Longrightarrow \lambda = 1, -2i, 2i.$$

$\underline{\lambda_1 = 1}$:

$$A - I = \begin{bmatrix} 1 & 0 & -4 \\ 0 & 0 & 0 \\ 2 & 0 & -3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -4 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$(A-I)x = 0 \Longleftrightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \Longrightarrow v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

$\underline{\lambda_2 = -2i:}$

$$A + 2iI = \begin{bmatrix} 2+2i & 0 & -4 \\ 0 & 1+2i & 0 \\ 2 & 0 & -2+2i \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1+i \\ 0 & 1+2i & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$(A+2iI)x = 0 \Longleftrightarrow x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1-i \\ 0 \\ 1 \end{bmatrix} \Longrightarrow v_2 = \begin{bmatrix} 1-i \\ 0 \\ 1 \end{bmatrix}.$$

$\underline{\lambda_3 = 2i:}$

Since A is a real matrix and $\lambda_3 = \overline{\lambda_2}$, an eigenvector for λ_3 is

$$v_3 = \overline{v_2} = \begin{bmatrix} 1+i \\ 0 \\ 1 \end{bmatrix}.$$

$A = PDP^{-1}$, where $P = \begin{bmatrix} 0 & 1-i & 1+i \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2i & 0 \\ 0 & 0 & 2i \end{bmatrix}$.

Homework: Let $A = \begin{bmatrix} -5 & 6 & 2 \\ -3 & 4 & 1 \\ -5 & 5 & 2 \end{bmatrix}$.

Find the eigenvalues and corresponding eigenvectors of A.

Answer: The eigenvalues are $\lambda_1 = 1$, $\lambda_2 = i$, $\lambda_3 = -i$
with corresponding eigenvalues

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 2 \\ 1 \\ 2-i \end{bmatrix}, v_3 = \begin{bmatrix} 2 \\ 1 \\ 2+i \end{bmatrix}.$$